On Symmetric and Upwind TVD Schemes

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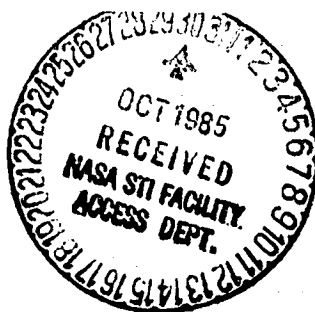
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I. Summary

A class of explicit and implicit total variation diminishing (TVD) schemes for the compressible Euler and Navier-Stokes equations has been developed [1-4]. They have the property of not generating spurious oscillations across shocks and contact discontinuities. In general, shocks can be captured within 1-2 grid points. For the inviscid case, one can divide these schemes into upwind TVD schemes and symmetric (non-upwind) TVD schemes. The upwind TVD scheme is based on the second-order TVD scheme developed by Harten [5]. The symmetric TVD scheme developed by the author is a generalization of Roe [6] and Davis's [7] TVD Lax-Wendroff scheme. The objective of this paper is to investigate the performance of these schemes on some viscous and inviscid airfoil steady-state calculations. A comparison of the symmetric and upwind TVD schemes is included.

II. Description of Algorithm for System of Hyperbolic Conservation Laws

The notion of upwind and symmetric TVD schemes, including formulation and extension to system cases (in uniform Cartesian grids), can be found in references [1-3,5]. Here the extension of the implicit second-order-accurate TVD scheme for hyperbolic systems of conservation laws in curvilinear coordinates [4] is briefly described.

Consider a two-dimensional system of hyperbolic conservation laws

$$\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} + \frac{\partial G(Q)}{\partial y} = 0. \tag{2.1}$$

Here Q, $F(Q)$ and $G(Q)$ are column vectors of m components.

A generalized coordinate transformation of the form $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ which maintains the strong conservation law form of equation (2.1) is given by

$$\frac{\partial \hat{Q}}{\partial t} + \frac{\partial \hat{F}(\hat{Q})}{\partial \xi} + \frac{\partial \hat{G}(\hat{Q})}{\partial \eta} = 0, \tag{2.2}$$

where $\hat{Q} = Q/J$, $\hat{F} = (\xi_x F + \xi_y G)/J$, $\hat{G} = (\eta_x F + \eta_y G)/J$, and $J = \xi_x \eta_y - \xi_y \eta_x$, the Jacobian transformation. Let $A = \partial F/\partial Q$ and $B = \partial G/\partial Q$; then the Jacobians $\hat{A}$ and $\hat{B}$ of $\hat{F}$ and $\hat{G}$ can be written as

$$\hat{A} = (\xi_x A + \xi_y B) \tag{2.3a}$$

$$\hat{B} = (\eta_x A + \eta_y B). \tag{2.3b}$$

Let the eigenvalues of $\hat{A}$ be $(a_\xi^1, a_\xi^2, ..., a_\xi^m)$ and the eigenvalues of $\hat{B}$ be $(a_\eta^1, a_\eta^2, ..., a_\eta^m)$. Denote R_ξ and R_η as the matrices whose columns are eigenvectors of $\hat{A}$ and $\hat{B}$, and denote R_ξ^{-1} and R_η^{-1} as the inverses of R_ξ and R_η.

Let the grid spacing be denoted by $\Delta\xi$ and $\Delta\eta$ such that $\xi = j\Delta\xi$ and $\eta = k\Delta\eta$. Let $Q_{j+\frac{1}{2},k}$ denote some symmetric average of $Q_{j,k}$ and $Q_{j+1,k}$ (for example, $Q_{j+\frac{1}{2},k} = 0.5*(Q_{j+1,k}+Q_{j,k})$,

or the Roe's average [8] for gas dynamics). Let $a_{j+\frac{1}{2}}^l$, $R_{j+\frac{1}{2}}$, $R_{j+\frac{1}{2}}^{-1}$ denote the quantities a_ξ^l, R_ξ, R_ξ^{-1} related to $\hat{A}$ evaluated at $Q_{j+\frac{1}{2},k}$. Similarly, let $a_{k+\frac{1}{2}}^l$, $R_{k+\frac{1}{2}}$, $R_{k+\frac{1}{2}}^{-1}$ denote the quantities a_η^l, R_η, R_η^{-1} related to $\hat{B}$ evaluated at $Q_{j,k+\frac{1}{2}}$.

Define

$$\alpha_{j+\frac{1}{2}} = R_{j+\frac{1}{2}}^{-1} \frac{Q_{j+1,k} - Q_{j,k}}{0.5 * (J_{j+1,k} + J_{j,k})} \tag{2.4a}$$

as the difference of the characteristic variables in the local ξ-direction, and define

$$\alpha_{k+\frac{1}{2}} = R_{k+\frac{1}{2}}^{-1} \frac{Q_{j,k+1} - Q_{j,k}}{0.5 * (J_{j,k+1} + J_{j,k})} \tag{2.4b}$$

as the difference of the characteristic variables in the local η-direction. The $J_{j,k}$ is the Jacobian transformation evaluated at $(j\Delta\xi, k\Delta\eta)$. The averaged Jacobians are used here in order to preserve freestream.

With the above notation, a one-parameter family of TVD schemes can be written as

$$\hat{Q}_{j,k}^{n+1} + \lambda^\xi \theta \left[\tilde{F}_{j+\frac{1}{2},k}^{n+1} - \tilde{F}_{j-\frac{1}{2},k}^{n+1} \right] + \lambda^\eta \theta \left[\tilde{G}_{j,k+\frac{1}{2}}^{n+1} - \tilde{G}_{j,k-\frac{1}{2}}^{n+1} \right]$$

$$= \hat{Q}_{j,k}^n - \lambda^\xi (1 - \theta) \left[\tilde{F}_{j+\frac{1}{2},k}^n - \tilde{F}_{j-\frac{1}{2},k}^n \right] - \lambda^\eta (1 - \theta) \left[\tilde{G}_{j,k+\frac{1}{2}}^n - \tilde{G}_{j,k-\frac{1}{2}}^n \right], \tag{2.5a}$$

where $0 \le \theta \le 1$, $\lambda^\xi = \Delta t/\Delta\xi$, $\lambda^\eta = \Delta t/\Delta\eta$, and Δt is the time step. This one-parameter family of schemes contains implicit as well as explicit schemes. When $\theta = 0$, (2.5) is an explicit method; when $\theta \ne 0$, it is an implicit scheme. For example, if $\theta = 1/2$, the time differencing is the trapezoidal formula and scheme (2.5) is second-order in time and space. If $\theta = 1$, the time differencing is the backward Euler method and scheme (2.5) is first-order in time but second-order in space.

The numerical flux function $\tilde{F}_{j+\frac{1}{2},k}$ for both the upwind and symmetric TVD schemes [2,3] can be expressed as

$$\tilde{F}_{j+\frac{1}{2},k} = \frac{1}{2} \left[\hat{F}_{j,k} + \hat{F}_{j+1,k} + R_{j+\frac{1}{2}} \Phi_{j+\frac{1}{2}} \right]. \tag{2.5b}$$

Similarly, we can define the numerical flux $\tilde{G}_{j,k+\frac{1}{2}}$.

Upwind TVD Scheme [5]

For a particular form of the upwind TVD scheme [2,5], the elements of the $\Phi_{j+\frac{1}{2}}$ denoted by $(\phi_{j+\frac{1}{2}}^l)^U$, $l = 1,...,m$ are

$$(\phi_{j+\frac{1}{2}}^l)^U - \frac{1}{2}\psi(a_{j+\frac{1}{2}}^l)(g_j^l + g_{j+1}^l) - \psi(a_{j+\frac{1}{2}}^l + \gamma_{j+\frac{1}{2}}^l)\alpha_{j+\frac{1}{2}}^l \tag{2.6a}$$

with

$$g_j^l = \text{minmod}(\alpha_{j-\frac{1}{2}}^l, \alpha_{j+\frac{1}{2}}^l). \tag{2.6b}$$

The minmod function of a list of arguments is equal to the smallest number in absolute value if the list of arguments is of the same sign, or is equal to zero if any argument is of opposite sign. The function ψ is

$$\psi(z) = \begin{cases} |z| & |z| \ge \epsilon \\ (z^2 + \epsilon^2)/2\epsilon & |z| < \epsilon \end{cases}. \tag{2.6c}$$

Here ϵ is a small positive parameter (see reference 5 for a formula of ϵ), and

$$\gamma_{j+\frac{1}{2}}^l = \frac{1}{2}\psi(a_{j+\frac{1}{2}}^l) \begin{cases} (g_{j+1}^l - g_j^l)/\alpha_{j+\frac{1}{2}}^l & \alpha_{j+\frac{1}{2}}^l \ne 0 \\ 0 & \alpha_{j+\frac{1}{2}}^l = 0, \end{cases} \tag{2.6d}$$

2

where $\alpha^l_{j+\frac{1}{2}}$ are elements of (2.4a).

Symmetric TVD Scheme [3]

For two particular forms of the symmetric TVD scheme, the elements of the $\Phi_{j+\frac{1}{2}}$ denoted by $(\phi^l_{j+\frac{1}{2}})^S$ are

$$(\phi^l_{j+\frac{1}{2}})^S = -\psi(a^l_{j+\frac{1}{2}})(2\alpha^l_{j+\frac{1}{2}} - g^l_j - g^l_{j+1}) \tag{2.7a}$$

and

$$(\phi^l_{j+\frac{1}{2}})^S = -\psi(a^l_{j+\frac{1}{2}})\left[\alpha^l_{j+\frac{1}{2}} - \text{minmod}(\alpha^l_{j-\frac{1}{2}}, \alpha^l_{j+\frac{1}{2}}, \alpha^l_{j+\frac{3}{2}})\right]. \tag{2.7b}$$

A Conservative Linearized ADI Form For Steady-State Application

For two-dimensional steady-state applications, the implicit schemes (2.5) can be solved by some appropriate relaxation method other than alternating direction implicit (ADI) and will be the direction of future research. The schemes considered in this paper are implemented in a conservative noniterative ADI form [2]. For steady-state applications, the numerical solution is independent of the time step. The implicit operator has a regular block tridiagonal structure and the resulting block tridiagonal matrix is diagonally dominant. One can modify a standard central difference code by simply changing the conventional numerical dissipation term into the one designed for the TVD scheme; i.e., the third term of equation (2.5b). The only difference in computation is that the current scheme requires a more elaborate dissipation term for the explicit operator; no extra computation is required for the implicit operator. For the Navier-Stokes applications, the convection terms are discretized by a TVD scheme, and the diffusion terms are discretized by central difference approximations.

A conservative linearized ADI form of equation (2.5) used mainly for steady-state applications as described in details in reference [2] can be written as

$$\left[I + \lambda^\xi \theta H^\xi_{j+\frac{1}{2},k} - \lambda^\xi \theta H^\xi_{j-\frac{1}{2},k}\right] D^* = -\lambda^\xi \left[\tilde{F}^n_{j+\frac{1}{2},k} - \tilde{F}^n_{j-\frac{1}{2},k}\right] - \lambda^\eta \left[\tilde{G}^n_{j,k+\frac{1}{2}} - \tilde{G}^n_{j,k-\frac{1}{2}}\right] \tag{3.1a}$$

$$\left[I + \lambda^\eta \theta H^\eta_{j,k+\frac{1}{2}} - \lambda^\eta \theta H^\eta_{j,k-\frac{1}{2}}\right] D = D^* \tag{3.1b}$$

$$\hat{Q}^{n+1} = \hat{Q}^n + D \tag{3.1c}$$

where

$$H^\xi_{j+\frac{1}{2},k} = \frac{1}{2}\left[\hat{A}_{j+1,k} + \Omega^\xi_{j+\frac{1}{2},k}\right]^n \tag{3.1d}$$

$$H^\eta_{j,k+\frac{1}{2}} = \frac{1}{2}\left[\hat{B}_{j,k+1} + \Omega^\eta_{j,k+\frac{1}{2}}\right]^n. \tag{3.1e}$$

The nonstandard notation

$$H^\xi_{j+\frac{1}{2},k} D^* = \frac{1}{2}\left[\hat{A}^n_{j+1,k} D^*_{j+1,k} + \Omega^\xi_{j+\frac{1}{2},k} D^*\right]^n \tag{3.1f}$$

is used and

$$\Omega^\xi_{j+\frac{1}{2},k} D^* = \text{diag}\left[-\max_l \psi(a^l_{j+\frac{1}{2}})\right](D^*_{j+1,k} - D^*_{j,k}) \tag{3.1g}$$

$$\Omega^\eta_{j,k+\frac{1}{2}} D = \text{diag}\left[-\max_l \psi(a^l_{k+\frac{1}{2}})\right](D_{j,k+1} - D_{j,k}). \tag{3.1h}$$

Here $\hat{A}_{j+1,k}$, $\hat{B}_{j,k+1}$ are (2.3) evaluated at $(j+1,k)$ and $(j,k+1)$. The expression diag(z^l) denotes a diagonal matrix with diagonal elements z^l. All of the inviscid calculations shown in this paper use (3.1).

3

For steady-state application, a simple algorithm utilizing the TVD scheme for the Navier-Stokes equations is to difference the hyperbolic terms the same way as before, and then central difference the viscous terms. The final algorithm is the same as equation (3.1) except that the spatial central differencing of the viscous term is added to the right hand side of equation (3.1). The numerical solution shown below illustrates that this algorithm produces a fairly good solution for the case of a RAE2822 airfoil calculation. A treatment for time-accurate calculations can be found in reference [2].

Numerical Results

Generally, for inviscid calculations, upwind TVD schemes produce sharper shocks than symmetric TVD schemes [7]. For the current two methods (2.6) and (2.7), this seems to be not the case. The two methods appeared to produce almost identical results for flow field conditions ranging from subcritical to transonic and supersonic. Figures 1 and 2 show the comparison of equation (2.6) with (2.7b) for two inviscid steady-state airfoil calculations. The advantages of symmetric TVD schemes are that they require less computational effort and provide a more natural way of extending the scheme to two and three-dimensional problems.

Figures 3 and 4 show an inviscid comparison of the symmetric TVD scheme with the widely distributed computer code ARC2D, version 150 [9]. The freestream Mach numbers are $M_\infty = 1.2$ and 1.8, and the angle of attack is $\alpha = 7$. The pressure coefficient distributions (not shown) are identical between the two methods and yet the flow field appears very different. The symmetric TVD scheme gives a very well-ordered flow structure and can still capture the shocks with a coarse grid, especially near the trailing edge of the airfoil. On the other hand, the ARC2D code did rather poorly. The same problem was studied for the upwind TVD scheme and the results and convergence rates were found to be almost identical to those for the symmetric TVD scheme. A residual of 10^{-12} can be reached at around 400-500 steps. ARC2D, however, required only 200-300 steps to converge to the same residual. The ARC2D, version 150 computer code is based on the Beam and Warming ADI algorithm [10] but uses a mixture of second and fourth-order numerical dissipation terms. Figure 5 is an example of the viscous case for the RAE2822 airfoil using the upwind TVD scheme. The thin layer Navier-Stokes equations with the algebraic turbulence model of Baldwin and Lomax [12] are used and the transition is fixed at 3% chord. The overall agreement with experiments is quite good. The L_2-norm residual of 10^{-7} can be reached in around 900 steps.

Concluding Remarks

Both the symmetric and upwind TVD schemes are designed to capture shock waves accurately while not exhibiting the spurious oscillation associated with the more classical second-order schemes. Numerical experiments with the TVD schemes on problems containing no shock [11] show that there is no advantage of the TVD scheme over the conventional Lax-Wendroff type scheme. Numerical experiments with various inviscid airfoil calculations show that for weak to moderate shocks the main advantage of the TVD schemes is that one can capture the shocks in 1-2 grid points [5]. The flow field away from the shock looks very much like the classical second-order central difference methods. But as one increases the shock strength, especially in supersonic flow, TVD schemes provide superior flow field solutions even with a very coarse yet non-clustering grid near shock waves. Numerical experiments also indicate that the symmetric and upwind TVD schemes are also applicable for viscous calculations. The current study further shows that the symmetric TVD scheme is just as accurate as the upwind TVD scheme while requiring less computational effort than the upwind TVD scheme.

References

[1] H.C. Yee, R.F. Warming and A. Harten, "Implicit Total Variation Diminishing (TVD) Schemes for Steady-State Calculations," AIAA Paper No. 83-1902, July, 1983.

[2] H.C. Yee, "Linearized Forms of Implicit TVD Schemes for the Multidimensional Euler and

Navier-Stokes Equations," International Journal on Computers and Mathematics with Applications, December 1985.

[3] H.C. Yee, "Generalized Formulation of a Class of Explicit and Implicit TVD Schemes," NASA-TM 86775, July 1985.

[4] H.C. Yee and A. Harten, "Implicit TVD Schemes for Hyperbolic Conservation Laws in Curvilinear Coordinates," AIAA Paper No. 85-1513, July 1985.

[5] A. Harten, "On a Class of High Resolution Total-Variation-Stable Finite-Difference Schemes," NYU Report, Oct., 1982; SIAM J. Num. Anal, Vol. 21, 1984, pp. 1-23.

[6] P.L. Roe, "Generalized Formulation of TVD Lax-Wendroff Schemes," ICASE Report No. 84-53, October 1984.

[7] S.F. Davis, "TVD Finite Difference Schemes and Artificial Viscosity," ICASE Report No. 84-20, June 1984.

[8] P.L. Roe, "Approximate Riemann Solvers, Parameter Vectors, and Difference Schemes," J. Comp. Phys. Vol. 43, 1981, pp. 357-372.

[9] T.H. Pulliam and J. Steger, "Recent Improvements in Efficiency, Accuracy and Convergence for Implicit Approximate Factorization Algorithms," AIAA Paper No. 85-0360, 1985.

[10] R.M. Beam and R.F. Warming, " An Implicit Finite-Difference Algorithm for Hyperbolic Systems in Conservation Law Form," J. Comp. Phys. Vol. 22, 1976, pp. 87-110.

[11] H.C. Yee, R.F. Warming and A. Harten, "Application of TVD Schemes for the Euler Equations of Gas Dynamics," Lectures in Applied Mathematics, Vol. 22, 1985.

[12] B. Baldwin and H. Lomax, "Thin Layer Approximation and Algebraic Model for Separated Turbulent Flows," AIAA Paper No. 78-257, 1978.
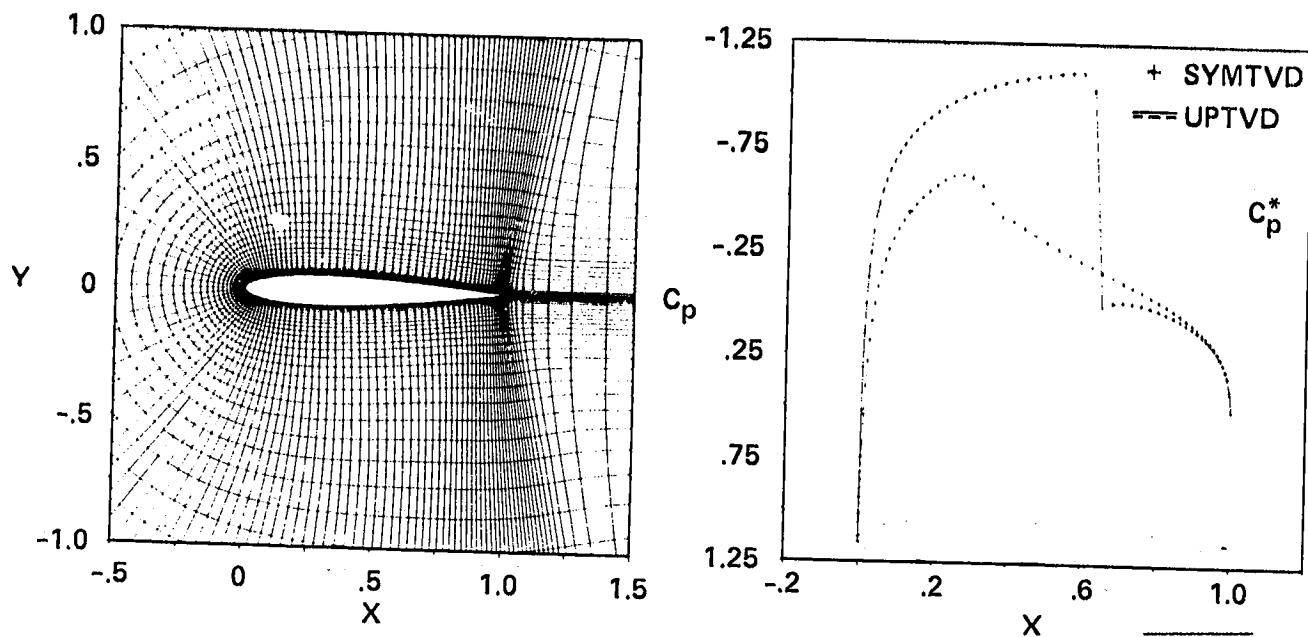
Fig. 1 Comparison of a symmetric TVD (SYMTVD) scheme with an upwind TVD (UPTVD) scheme for the NACA0012 airfoil with $M_\infty = 0.8$, $\alpha = 1.25$ using a 163×49 C grid.
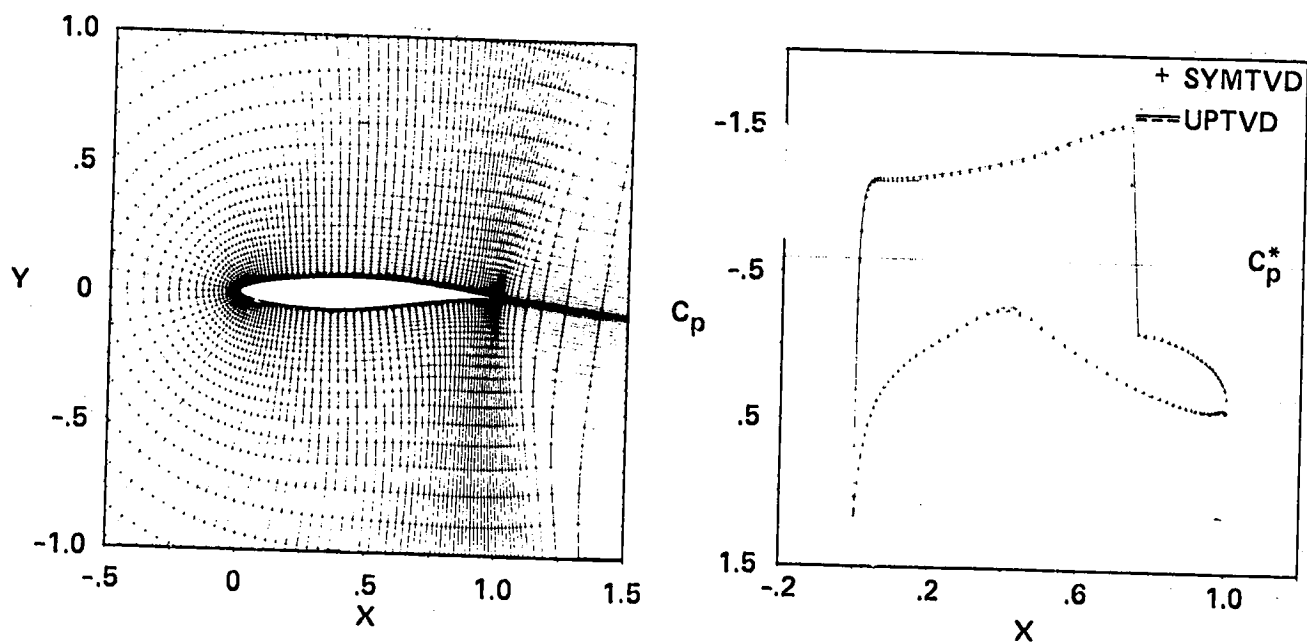

Fig. 2 Comparison of a symmetric TVD (SYMTVD) scheme with an upwind TVD (UPTVD) scheme for the RAE2822 airfoil with $M_\infty = 0.75$, $\alpha = 3$ using a 163×49 C grid.
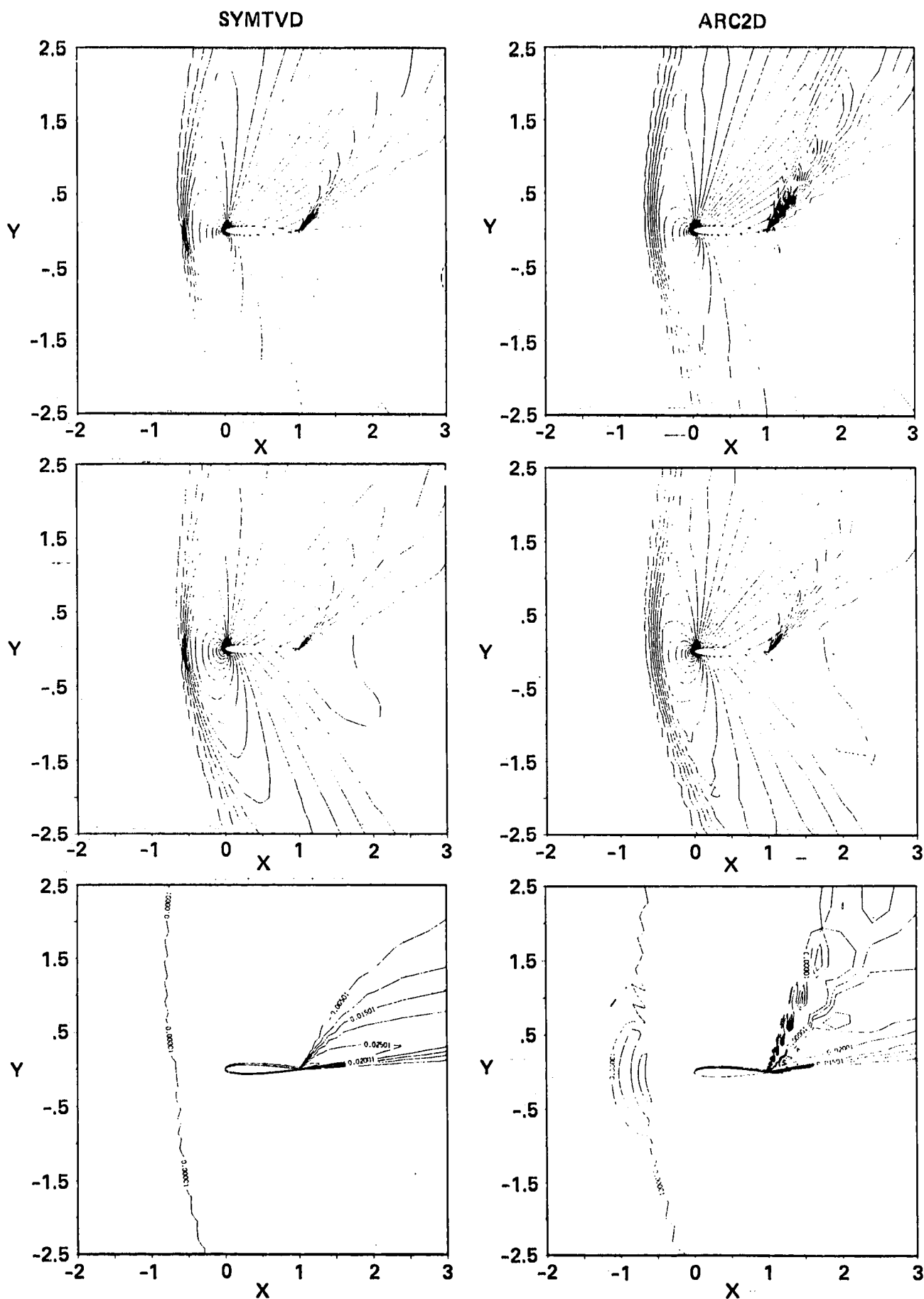
6

Fig. 3 Comparison of a symmetric TVD (SYMTVD) scheme with ARC2D (version 150) for the Mach contours, pressure contours and entropy contours of the NACA0012 airfoil with $M_\infty = 1.2$, $\alpha = 7$ using a 163×49 C grid as shown in figure 1.
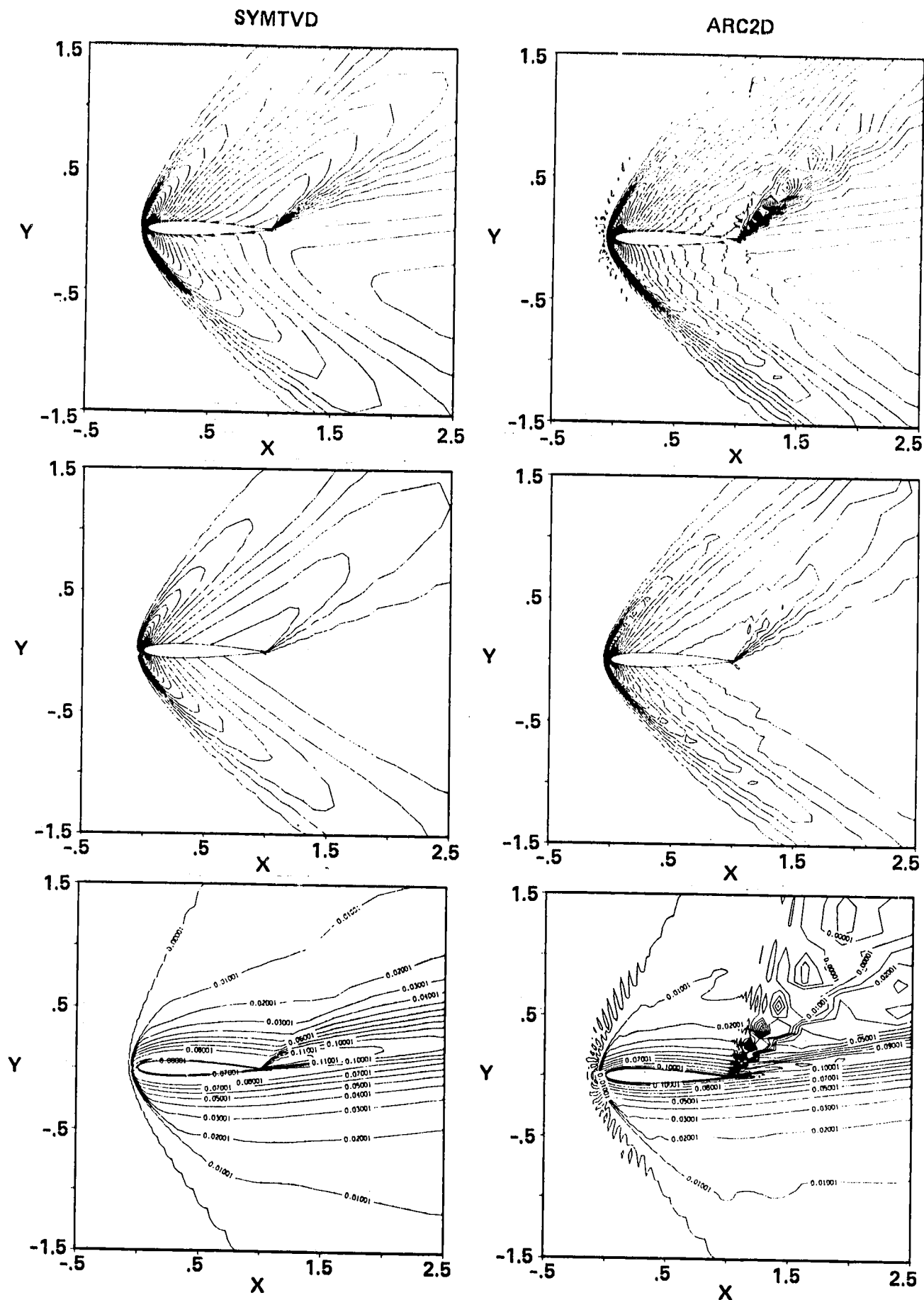
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SYMTVD

ARC2D

Fig. 4 Comparison of a symmetric TVD (SYMTVD) scheme with ARC2D (version 150) for the Mach contours, pressure contours and entropy contours of the NACA0012 airfoil with $M_\infty = 1.8$, $\alpha = 7$ using a 163 × 49 C grid as shown in figure 1.

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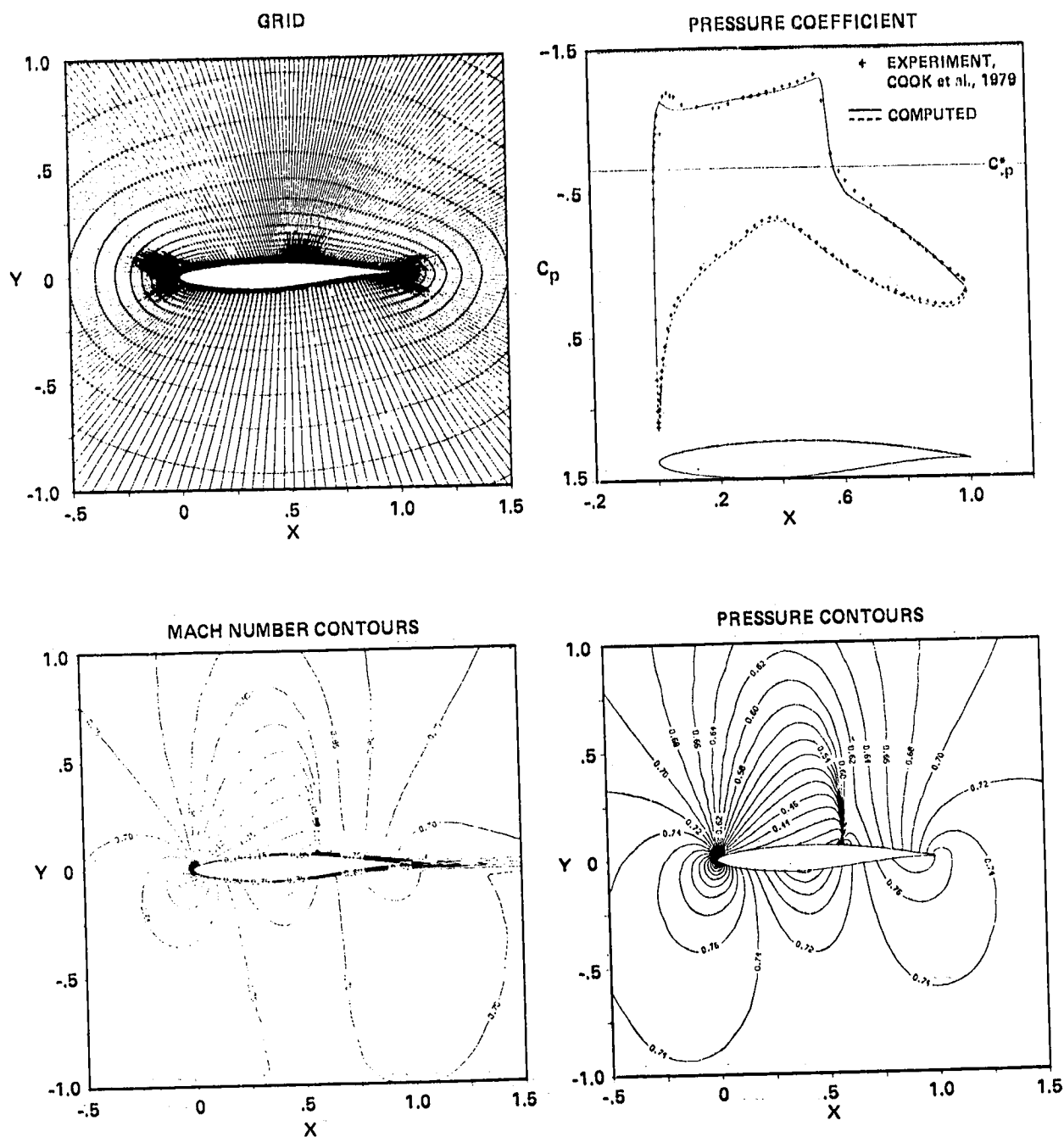GRID

PRESSURE COEFFICIENT

MACH NUMBER CONTOURS

PRESSURE CONTOURS

Fig. 5 A thin layer Navier-Stokes calculation for the RAE2822 airfoil using an upwind TVD scheme with a
249 × 51 O grid, $M_\infty = 0.73$, $\alpha = 2.79$, $Re = 6.5 \times 10^6$.

9

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16. Abstract

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